



TITLE:

# Algebraic independence results for Ramanujan $q$ -series (New Aspects of Analytic Number Theory)

AUTHOR(S):

Elsner, Carsten; Shimomura, Shun; Shiokawa,  
Iekata

---

CITATION:

Elsner, Carsten ...[et al]. Algebraic independence results for Ramanujan  $q$ -series (New Aspects of Analytic Number Theory). 数理解析研究所講究録 2009, 1639: 172-175

ISSUE DATE:

2009-04

URL:

<http://hdl.handle.net/2433/140546>

RIGHT:

# Algebraic independence results for Ramanujan $q$ -series

Carsten Elsner\*

Fachhochschule für die Wirtschaft, University of Applied Sciences, Freundallee 15 D-30173 Hannover, Germany

Shun Shimomura†, Iekata Shiokawa‡

Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

## 1 Introduction

In 1916, Ramanujan [8] defined the function

$$S_{2j+1}(x) = \frac{1}{2}\zeta(-2j-1) + \sum_{n=1}^{\infty} \frac{n^{2j+1}x^n}{1-x^n} \quad (j = 0, 1, 2, \dots),$$

where  $\zeta(s)$  is the Riemann zeta function, in particular

$$P(x) = -24S_1(x), \quad Q(x) = 240S_3(x), \quad R(x) = -540S_5(x),$$

and proved various formulas for these functions.

Nesterenko [7] proved that, for an algebraic number  $x$  with  $0 < |x| < 1$ , the three numbers  $P(x)$ ,  $Q(x)$ ,  $R(x)$  are algebraically independent.

In this paper we study, for a given algebraic number  $x$  with  $0 < |x| < 1$ , the algebraic independence of the numbers  $S_{2j+1}(x)$  ( $j = 0, 1, 2, \dots$ ), or equivalently that of the numbers

$$A_{2j+1} = A_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{n^{2j+1}q^{2n}}{1-q^{2n}} \quad (j = 0, 1, 2, \dots) \quad (1)$$

for  $q^2 = x$  (using the symbol in [10]). We remark that the  $q$ -series identity

$$A_7(q) = A_3(q) + 120A_3(q)^2 \quad (2)$$

follows from Table I in [8]. Our result is stated as follows:

**Theorem 1.** *The  $q$ -series  $A_{2i+1}(q)$  and  $A_{2j+1}(q)$  with  $1 \leq i < j$  are algebraically dependent over  $\overline{\mathbb{Q}}(q)$  if and only if  $(i, j) = (1, 3)$ .*

**Theorem 2.** *Let  $q$  be an algebraic number with  $0 < |q| < 1$ . Then the three numbers  $A_1(q)$ ,  $A_{2i+1}(q)$  and  $A_{2j+1}(q)$  with  $1 \leq i < j$  and  $(i, j) \neq (1, 3)$  are algebraically independent.*

As an immediate corollary of Theorem 2 we have

**Corollary 1.** *The  $q$ -series  $A_1(q)$ ,  $A_{2i+1}(q)$  and  $A_{2j+1}(q)$  with  $1 \leq i < j$  and  $(i, j) \neq (1, 3)$  are algebraically independent over  $\overline{\mathbb{Q}}(q)$ .*

---

\*carsten.elsner@fhwd.de

†shimomur@math.keio.ac.jp

‡shiokawa@beige.ocn.ne.jp

It is easy to see that, for any positive integers  $i, j, l$ , the  $q$ -series  $A_{2i+1}(q)$ ,  $A_{2j+1}(q)$ ,  $A_{2l+1}(q)$  are algebraically dependent over  $\mathbb{Q}$  (cf. Lemmas 2 and 3 in Section 2). Our proof of the theorem depends on three basic tools; namely, a corollary of Nesterenko's theorem (see Lemma 1), expressions of  $A_{2j+1}(q)$  in terms of  $K/\pi$ ,  $E/\pi$  and  $k$  given by Ramanujan [8], Zucker [10] and the authors [2], where  $K$  and  $E$  are the complete elliptic integrals of the first and second kind with the modulus  $k$  (see Lemmas 2 and 3), and an algebraic independence criterion (see Lemma 6). Our method of this paper can be applied to certain sequences of  $q$ -series studied by Zucker [10] and the authors [2], [3], [4].

## 2 Lemmas

Let  $K$  and  $E$  denote the complete elliptic integrals of the first and second kind

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad E = E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}} dt,$$

with the modulus  $k$  such that  $k^2 \in \mathbb{C} \setminus (\{0\} \cup [1, +\infty))$ . The branch of each integrand is chosen so that it tends to 1 as  $t \rightarrow 0$ . Moreover set  $K' = K(k')$ ,  $k^2 + (k')^2 = 1$ . For any  $q \in \mathbb{C}$  with  $0 < |q| < 1$ , choose  $K'(k)/K(k)$  so that  $q = e^{-\pi K'/K}$ . Then by Nesterenko's theorem ([7], see also [2, §3]) we have

**Lemma 1.** *If  $q \in \overline{\mathbb{Q}}$  with  $0 < |q| < 1$ , then  $k$ ,  $K/\pi$ , and  $E/\pi$  are algebraically independent.*

The Jacobian elliptic function  $w = \operatorname{sn} z = \operatorname{sn}(z, k)$  may be defined by

$$z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

and we write

$$\operatorname{ns} z = \operatorname{ns}(z, k) = \frac{1}{\operatorname{sn}(z, k)}, \quad \operatorname{cs} z = \operatorname{cs}(z, k) = \frac{\operatorname{cn}(z, k)}{\operatorname{sn}(z, k)}, \quad \operatorname{cn}(z, k) = \sqrt{1 - \operatorname{sn}^2(z, k)}.$$

Let  $a_j$  and  $c_j = c_j(k)$  ( $j \geq 0$ ) denote the coefficients of the series expansions

$$\operatorname{cosec}^2 z = z^{-2} + \sum_{j=0}^{\infty} a_j z^{2j}, \quad \operatorname{ns}^2(z, k) = z^{-2} + \sum_{j=0}^{\infty} c_j(k) z^{2j}, \quad (3)$$

respectively; in particular

$$a_j := \frac{(-1)^j (2j+1) 2^{2j+2} B_{2j+2}}{(2j+2)!} \quad (j \geq 0) \quad (4)$$

with the Bernoulli numbers  $B_2 = 1/6$ ,  $B_4 = -1/30$ ,  $B_6 = 1/42$ ,  $B_8 = -1/30$ , ... etc. The following evaluations of  $A_{2j+1}(q)$  in terms of  $k$ ,  $K/\pi$  and  $E/\pi$  were given by Ramanujan [8] for  $j = 1, 2, \dots, 15$  and completed by Zucker [10].

**Lemma 2.** *For  $q = e^{-\pi K'/K}$ , the  $q$ -series  $A_{2j+1}(q)$  is expressed in the form*

$$A_1(q) = \frac{1}{24} - \frac{1}{24} \left( \frac{2K}{\pi} \right)^2 \left( \frac{3E}{K} - 2 + k^2 \right),$$

$$A_{2j+1}(q) = (-1)^j \frac{(2j)!}{2^{2j+3}} \left( a_j - \left( \frac{2K}{\pi} \right)^{2j+2} c_j(k) \right) \quad (j \geq 1).$$

The coefficients of the series for  $\operatorname{ns}^2(z, k)$  are determined recursively ([2, Lemma 2]):

**Lemma 3.** The coefficients  $c_j = c_j(k)$  are given by

$$c_0 = \frac{1}{3}(1+k^2), \quad c_1 = \frac{1}{15}(1-k^2+k^4), \quad c_2 = \frac{1}{189}(1+k^2)(1-2k^2)(2-k^2),$$

$$(j-2)(2j+3)c_j = 3 \sum_{i=1}^{j-2} c_i c_{j-i-1} \quad (j \geq 3).$$

It follows from this lemma that  $c_j = c_j(k) \in \mathbb{Q}[k^2]$  with  $\deg c_j(k) = 2(j+1)$ . Put

$$c_j(k) = \alpha_{j,0} + \alpha_{j,1}k^2 + \dots + \alpha_{j,j+1}k^{2j+2} \quad (j \geq 1). \quad (5)$$

It is easy to see that  $z^2 ns^2(z, k)$  is analytic around  $(z, k) = (0, 0)$ , and that  $z^2 ns^2(z, k) = 1 + O(|z|^2 + |k|^2)$ . The following lemma gives a detailed expression.

**Lemma 4.** Around  $(z, k) = (0, 0)$  we have

$$ns^2(z, k) = \operatorname{cosec}^2 z + f(z)k^2 + g(z)k^4 + O(k^6),$$

with

$$f(z) = \frac{z}{4}(\cot z)'' + \frac{1}{2}(\cot z)' + \frac{1}{2} = \frac{1}{2} - \sum_{j=0}^{\infty} \frac{(-1)^{j+1} 2^{2j} B_{2j+2}}{(2j)!} z^{2j},$$

$$g(z) = -\frac{z^2}{32}(\cot z)^{(3)} - \frac{3z}{64}(\cot z)'' + \frac{3}{32}(\cot z)' + \frac{1}{16} - \frac{1}{32} \cos 2z$$

$$= \frac{1}{16} + \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j-5}}{(2j)!} (2(4j-3)B_{2j+2} - 1) z^{2j}.$$

From (3), (4), (5) and Lemma 4 we deduce the following corollary.

**Corollary 2.** For  $j \geq 1$  we have

$$\alpha_{j,0} = \frac{(-1)^j (2j+1) 2^{2j+2} B_{2j+2}}{(2j+2)!},$$

$$\alpha_{j,1} = \frac{(-1)^{j+1} 2^{2j} B_{2j+2}}{(2j)!} \quad (j \geq 1), \quad \alpha_{0,1} = \frac{1}{3},$$

$$\alpha_{j,2} = \frac{(-1)^j 2^{2j-5}}{(2j)!} (2(4j-3)B_{2j+2} - 1) \quad (j \geq 1), \quad \alpha_{0,2} = 0.$$

**Lemma 5.** All Bernoulli numbers  $B_{2n}$  are distinct with the only exception  $B_4 = B_8 = -1/30$ .

**Lemma 6.** (cf. [5]) Let  $x_1, \dots, x_n \in \mathbb{C}$  be algebraically independent and let  $y_j := U_j(x_1, \dots, x_n)$ , where  $U_j(X_1, \dots, X_n) \in \mathbb{Q}[X_1, \dots, X_n]$  ( $j = 1, \dots, n$ ). Assume that

$$\det \left( \frac{\partial U_j}{\partial X_i}(x_1, \dots, x_n) \right) \neq 0. \quad (6)$$

Then the numbers  $y_1, \dots, y_n$  are algebraically independent.

### 3 Sketch of the proof of Theorem 1

We assume that two  $q$ -series  $A_{2i+1}(q)$  and  $A_{2j+1}(q)$  with  $1 \leq i < j$  are algebraically dependent over  $\overline{\mathbb{Q}}(q)$  and deduce the case  $(i, j) = (1, 3)$ . There exists an algebraic number  $q_0$  with  $0 < |q_0| < 1$  such that the two numbers  $A_{2i+1}(q_0)$  and  $A_{2j+1}(q_0)$  are algebraically dependent. Choose  $k$  so that  $q_0 = e^{-\pi K'(k)/K(k)}$ . We apply Lemma 6 with Lemma 1 and 2 by putting  $x_1 = k$ ,  $x_2 = K/\pi$ ,  $y_\nu = U_\nu(x_1, x_2)$  ( $\nu = 1, 2$ ), where  $U_1(X_1, X_2) = X_2^{2i+2}c_i(X_1)$ ,  $U_2(X_1, X_2) = X_2^{2j+2}c_j(X_1)$ . Then the vanishing of the determinant implies

$$(j+1)c'_i(k)c_j(k) - (i+1)c'_j(k)c_i(k) = 0$$

as a polynomial of  $k$ , or equivalently  $c_j(k)^{i+1} = rc_i(k)^{j+1}$  for some  $r \in \mathbb{Q} \setminus \{0\}$ . Substituting (5) into both sides of the last identity and comparing the coefficients of  $k^2$  and  $k^4$ , we get

$$\frac{2\alpha_{j,0}\alpha_{j,2} + i\alpha_{j,1}^2}{\alpha_{j,0}\alpha_{j,1}} = \frac{2\alpha_{i,0}\alpha_{i,2} + j\alpha_{i,1}^2}{\alpha_{i,0}\alpha_{i,1}}.$$

This with the explicit values of  $\alpha_{i,j}$  given in Corollary 2 leads to

$$2(4j-3) - 1/B_{2j+2} + 8i(j+1) = 2(4i-3) - 1/B_{2i+2} + 8j(i+1),$$

that is  $B_{2i+2} = B_{2j+2}$ . By Lemma 5, we see that  $(i, j) = (1, 3)$ . The ‘if part’ follows from the formula (2), and the proof is completed.

Similarly, we can prove Theorem 2, from which Corollary 1 follows.

### References

- [1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1970.
- [2] C. Elsner, S. Shimomura and I. Shiokawa, *Algebraic relations for reciprocal sums of Fibonacci numbers*, Acta Arith. **130** (2007), 37–60.
- [3] C. Elsner, S. Shimomura and I. Shiokawa, *Algebraic relations for reciprocal sums of odd terms in Fibonacci numbers*, Ramanujan J. **17** (2008), 429–446.
- [4] C. Elsner, S. Shimomura and I. Shiokawa, *Algebraic relations for reciprocal sums of even terms in Fibonacci numbers*, to appear in Algebra i Analiz; English transl. St. Petersburg Math. J.
- [5] C. Elsner, S. Shimomura and I. Shiokawa, *Algebraic independence results for reciprocal sums of Fibonacci numbers*, submitted paper.
- [6] H. Hancock, *Theory of Elliptic Functions*, Dover, New York, 1958.
- [7] Yu. V. Nesterenko, *Modular functions and transcendence questions*, Mat. Sb. **187** (1996), 65–96; English transl. Sb. Math. **187** (1996), 1319–1348.
- [8] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), 159–184.
- [9] E. T. Whittaker and G. N. Watson, *Modern Analysis*, 4th ed. Cambridge Univ. Press, Cambridge, 1927.
- [10] I. J. Zucker, *The summation of series of hyperbolic functions*, SIAM J. Math. Anal. **10** (1979), 192–206.